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Compactness in $\mathcal{B}(X)$ ☆

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Abstract

This paper is concerned with compactness for some topologies on the collection of bounded linear operators on Banach spaces. New versions of the Eberlein–Šmulian theorem and Day's lemma in the collection are established. Also we obtain a partial solution of the dual problem for the quasi approximation property, that is, it is shown that for a Banach space X if X^{**} is separable and X^* has the quasi approximation property, then X has the quasi approximation property.

Keywords: Weak operator topology; Weak* operator topology; Strong operator topology; τ-topology; Quasi approximation property

1. Introduction and main results

In topological spaces, compactness is a fundamental property. Many mathematicians have obtained important results for compactness including Stefan Banach, Leonidas Alaoglu, Robert C. James, William F. Eberlein, and Vitold L. Šmulian who were interested in *weak* and *weak* compactness*. They proved:

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The Banach–Alaoglu Theorem. [1,2] *Let X be a normed space. Then the unit ball B_{X^*} in X^* is weak* compact.*

The James's Weak Compactness Theorem. [10] *Suppose that A is a weakly closed subset of a Banach space X. Then the following are equivalent.*

(a) *A is weakly compact.*
(b) *For every $x^* \in X^*$ the supremum of $|x^*|$ is attained on A.*

The Eberlein–Šmulian Theorem. [8,15] *Let A be a subset of a normed space. Then the following are equivalent.*

(a) *A is weakly compact.*
(b) *A is weakly countably compact.*
(c) *A is weakly limit point compact.*
(d) *A is weakly sequentially compact.*

For concrete proofs of above results one may see ([14, Theorems 2.6.18, 2.9.3, 2.8.6], [7, Theorems V.4.2, V.6.1], [6, pp. 13, 18]). The purpose of this paper is to obtain new versions of the Eberlein–Šmulian theorem for some topologies on the collection of bounded linear operators on separable Banach spaces and to give a new version of Day's lemma in the collection. Also we obtain applications of some weak versions of the approximation property.

Now we introduce some important topologies on the collection of bounded linear operators on Banach spaces. Throughout this paper we denote the collection of bounded linear operators on a Banach space X by $\mathcal{B}(X)$.

At first, we introduce two topologies generated by subspaces of the vector space of all linear functionals on $\mathcal{B}(X)$.

Definition 1.1. Let X be a Banach space. Let $\mathcal{Z}$ be the linear span of all linear functionals f on $\mathcal{B}(X)$ of the form $f(T) = x^*Tx$ for $x \in X$ and $x^* \in X^*$. Then the *weak operator topology* (in short, *wo*) on $\mathcal{B}(X)$ is the topology generated by $\mathcal{Z}$.

We can check that *wo* is a locally convex topology and the basic neighborhoods of *wo* is

$$B(T; A, B, \epsilon) = \{R \in \mathcal{B}(X)\colon\ |x^*Rx - x^*Tx| < \epsilon,\ x \in A,\ x^* \in B\},$$

where A and B are arbitrary finite sets in X and X^*, respectively, and $\epsilon > 0$ is arbitrary. We can easily check that *wo* is a T_0 space. So *wo* is completely regular since every T_0 vector topology is completely regular. Also for a net $(T_\alpha) \subset \mathcal{B}(X)$ and $T \in \mathcal{B}(X)$

$$T_\alpha \to T \quad \text{in } (\mathcal{B}(X), wo) \quad \Leftrightarrow \quad \text{for each } x \in X \text{ and } x^* \in X^* \quad x^*T_\alpha x \to x^*Tx. \tag{1.1}$$

We say that a topological space has the *second countability axiom* if the space has a countable base for the topology. For *wo* we have the following theorem (see Section 3 for proofs).

Theorem 1.2. *Let X be a Banach space such that X^* is separable. If $\mathcal{A}$ is a wo-bounded subset of $\mathcal{B}(X)$, then the relative wo-topology of $\mathcal{A}$ has the wo-second countability axiom and is metrizable.*

From Theorem 1.2 for separable case we get the following weak operator topology version of the Eberlein–Šmulian theorem.

Corollary 1.3. *Let X be a Banach space such that X^* is separable and let $\mathcal{A}$ be a subset of $\mathcal{B}(X)$. Then the following are equivalent.*

(a) *$\mathcal{A}$ is wo-compact.*
(b) *$\mathcal{A}$ is wo-countably compact.*
(c) *$\mathcal{A}$ is wo-limit point compact.*
(d) *$\mathcal{A}$ is wo-sequentially compact.*

Proof. Note that in any topological spaces compactness implies countable compactness, countable compactness implies limit point compactness, and sequential compactness implies limit point compactness. In Section 2 we will see that *wo*-limit point compact is bounded and so is *wo*-bounded. Hence by Theorem 1.2 we complete the proof. □

We say that a topological space is *Lindelöf* if every open covering for the space contains a countable subcovering. Note that the second countability axiom imply Lindelöf and separability. Hence by Theorem 1.2 we have the following conclusions.

Corollary 1.4. *Let X be a Banach space such that X^* is separable. Then $\mathcal{B}(X)$ is wo-separable.*

Proof. Let $B(0; n)$ be the ball in $\mathcal{B}(X)$ with center 0 and radius n for each n. Then by Theorem 1.2 ($B(0; n)$, the relative *wo*-topology of $B(0; n)$) has a countable dense subset $\{T_{n,m}\}_{m\in\mathbf{N}}$ for each n. Consider $\{T_{n,m}\}_{(n,m)\in\mathbf{N}\times\mathbf{N}}$. Then clearly $\{T_{n,m}\}_{(n,m)\in\mathbf{N}\times\mathbf{N}}$ is a *wo*-dense subset of $\mathcal{B}(X)$. Hence $\mathcal{B}(X)$ is *wo*-separable.

Corollary 1.5. *Let X be a Banach space such that X^* is separable. If $\mathcal{A}$ is a wo-bounded subset of $\mathcal{B}(X)$, then $\mathcal{A}$ is wo-Lindelöf.*

For nonseparable case we have:

Theorem 1.6. *Let X be a Banach space. If $\mathcal{A}$ is a relatively weakly compact subset of $\mathcal{B}(X)$ and $T \in \overline{\mathcal{A}}^{wo}$, then there is a sequence (T_n) in $\mathcal{A}$ such that (T_n) weakly converges to T.*

Definition 1.7. Let X be a Banach space. Let $\mathcal{Z}$ be the linear span of all linear functionals f on $\mathcal{B}(X)$ of the form $f(T) = x^{**}T^*x^*$ for $x^* \in X^*$ and $x^{**} \in X^{**}$. Then the *weak* operator topology* (in short, w^*o) on $\mathcal{B}(X)$ is the topology generated by $\mathcal{Z}$.

We can check that w^*o is a locally convex topology and the basic neighborhoods of w^*o is

$$B(T; A, B, \epsilon) = \{R \in \mathcal{B}(X): |x^{**}R^*x^* - x^{**}T^*x^*| < \epsilon,\ x^* \in A,\ x^{**} \in B\},$$

where A and B are arbitrary finite sets in X^* and X^{**}, respectively, and $\epsilon > 0$ is arbitrary. We can see $w^*o \geqslant wo$. So w^*o is a T_0 space since wo is a T_0 space. Thus w^*o is completely regular since every T_0 vector topology is completely regular. Also for a net $(T_\alpha) \subset \mathcal{B}(X)$ and $T \in \mathcal{B}(X)$

$$T_\alpha \to T \text{ in } (\mathcal{B}(X), w^*o) \quad \Leftrightarrow \quad \text{for each } x^* \in X^* \text{ and } x^{**} \in X^{**} \quad x^{**}T_\alpha^*x^* \to x^{**}T^*x^*. \tag{1.2}$$

For w^*o we have the following theorem.

Theorem 1.8. *Let X be a Banach space such that X^{**} is separable. If $\mathcal{A}$ is a w^*o-bounded subset of $\mathcal{B}(X)$, then the relative w^*o-topology of $\mathcal{A}$ has the w^*o-second countability axiom and is metrizable.*

As the weak operator topology we have the following corollaries.

Corollary 1.9. *Let X be a Banach space such that X^{**} is separable and let $\mathcal{A}$ be a subset of $\mathcal{B}(X)$. Then the following are equivalent.*

(a) *$\mathcal{A}$ is w^*o-compact.*
(b) *$\mathcal{A}$ is w^*o-countably compact.*
(c) *$\mathcal{A}$ is w^*o-limit point compact.*
(d) *$\mathcal{A}$ is w^*o-sequentially compact.*

Corollary 1.10. *Let X be a Banach space such that X^{**} is separable. Then $\mathcal{B}(X)$ is w^*o-separable.*

Corollary 1.11. *Let X be a Banach space such that X^{**} is separable. If $\mathcal{A}$ is a w^*o-bounded subset of $\mathcal{B}(X)$, then $\mathcal{A}$ is w^*o-Lindelöf.*

For some results for the weak operator topology and the weak* operator topology one may see Kalton [11].

Now we introduce two topologies on $\mathcal{B}(X)$ generated by some subbases. The following topology is also called the *topology of pointwise convergence*.

Definition 1.12. Let X be a Banach space. For $x \in X$, $\epsilon > 0$, and $T \in \mathcal{B}(X)$ we put

$$B(T, x, \epsilon) = \{R \in \mathcal{B}(X): \|Rx - Tx\| < \epsilon\}.$$

Let $\mathcal{S}$ be the collection of all such $B(T, x, \epsilon)$'s. Then the *strong operator topology* (in short, *sto*) on $\mathcal{B}(X)$ is the topology generated by $\mathcal{S}$.

We can check that *sto* is a locally convex topology and for a net $(T_\alpha) \subset \mathcal{B}(X)$ and $T \in \mathcal{B}(X)$

$$T_\alpha \to T \quad \text{in } (\mathcal{B}(X), sto) \quad \Leftrightarrow \quad \text{for each } x \in X \quad \|T_\alpha x - Tx\| \to 0. \tag{1.3}$$

We can easily check that *sto* is a T_0 space. So *sto* is completely regular since every T_0 vector topology is completely regular.

For *sto* we have the following theorem.

Theorem 1.13. *Suppose that X is a separable Banach space and let $\mathcal{A}$ be a sto-bounded subset of $\mathcal{B}(X)$. Then the relative sto-topology of $\mathcal{A}$ has the sto-second countability axiom and is metrizable.*

As the weak operator topology we have the following corollaries.

Corollary 1.14. *Suppose that X is a separable Banach space and let $\mathcal{A}$ be a subset of $\mathcal{B}(X)$. Then the following are equivalent.*

(a) *$\mathcal{A}$ is sto-compact.*
(b) *$\mathcal{A}$ is sto-countably compact.*
(c) *$\mathcal{A}$ is sto-limit point compact.*
(d) *$\mathcal{A}$ is sto-sequentially compact.*

Corollary 1.15. *Suppose that X is a separable Banach space. Then $\mathcal{B}(X)$ is sto-separable.*

Corollary 1.16. *Suppose that X is a separable Banach space. If $\mathcal{A}$ is a sto-bounded subset of $\mathcal{B}(X)$, then $\mathcal{A}$ is sto-Lindelöf.*

The following topology is also called the *topology of compact convergence*.

Definition 1.17. Let X be a Banach space. For compact $K \subset X$, $\epsilon > 0$, and $T \in \mathcal{B}(X)$ we put

$$B(T, K, \epsilon) = \Big\{ R \in \mathcal{B}(X): \sup_{x \in K} \|Rx - Tx\| < \epsilon \Big\}.$$

Let $\mathcal{S}$ be the collection of all such $B(T, K, \epsilon)$'s. Then the τ-topology (in short, τ) on $\mathcal{B}(X)$ is the topology generated by $\mathcal{S}$.

We can check that τ is a locally convex topology and for a net $(T_\alpha) \subset \mathcal{B}(X)$ and for $T \in \mathcal{B}(X)$

$$T_\alpha \to T \quad \text{in } (\mathcal{B}(X), \tau) \quad \Leftrightarrow \quad \text{for each compact } K \subset X \quad \sup_{x \in K} \|T_\alpha x - Tx\| \to 0. \tag{1.4}$$

We can see $\tau \geqslant sto$. So τ is a T_0 space since *sto* is a T_0 space. Thus τ is completely regular since every T_0 vector topology is completely regular.

For τ we have the following theorem.

Theorem 1.18. *Suppose that X is a separable Banach space and let $\mathcal{A}$ be a τ-bounded subset of $\mathcal{B}(X)$. Then the relative τ-topology of $\mathcal{A}$ has the τ-second countability axiom and is metrizable.*

As the weak operator topology we have the following corollaries.

Corollary 1.19. *Suppose that X is a separable Banach space and let $\mathcal{A}$ be a subset of $\mathcal{B}(X)$. Then the following are equivalent.*

(a) *$\mathcal{A}$ is τ-compact.*
(b) *$\mathcal{A}$ is τ-countably compact.*
(c) *$\mathcal{A}$ is τ-limit point compact.*
(d) *$\mathcal{A}$ is τ-sequentially compact.*

Corollary 1.20. *Suppose that X is a separable Banach space. Then $\mathcal{B}(X)$ is τ-separable.*

Corollary 1.21. *Suppose that X is a separable Banach space. If $\mathcal{A}$ is a τ-bounded subset of $\mathcal{B}(X)$, then $\mathcal{A}$ is τ-Lindelöf.*

Note that for a net $(T_\alpha) \subset \mathcal{B}(X)$ and $T \in \mathcal{B}(X)$

$$\begin{aligned} &T_\alpha \to T \quad \text{in } (\mathcal{B}(X),\ \textit{the weak topology}) \\ &\quad \Leftrightarrow \quad \text{for each } f \in (\mathcal{B}(X),\ \textit{the operator norm topology})^* \quad f(T_\alpha) \to f(T) \end{aligned}$$

and the *weak topology* on $\mathcal{B}(X)$ is completely regular. Thus by (1.1) and (1.2)

$$\textit{the operator norm topology} \geqslant \textit{the weak topology} \geqslant w^*o \geqslant wo. \tag{1.5}$$

Also by (1.1), (1.3), and (1.4)

$$\textit{the operator norm topology} \geqslant \tau \geqslant sto \geqslant wo. \tag{1.6}$$

From Theorem 1.6, (1.5), and (1.6) we have the following corollaries. Corollary 1.22 is the Day's lemma ([5, Theorem 3.2.4], [14, Lemma 2.8.5]) in $\mathcal{B}(X)$.

Corollary 1.22. *Let X be a Banach space. If $\mathcal{A}$ is a relatively weakly compact subset of $\mathcal{B}(X)$ and $T \in \overline{\mathcal{A}}^{weak} \subset \overline{\mathcal{A}}^{w^*o} \subset \overline{\mathcal{A}}^{wo}$, then there is a sequence (T_n) in $\mathcal{A}$ such that (T_n) weakly converges to T.*

Corollary 1.23. *Let X be a Banach space. If $\mathcal{A}$ is a relatively weakly compact subset of $\mathcal{B}(X)$ and $T \in \overline{\mathcal{A}}^{\tau} \subset \overline{\mathcal{A}}^{sto} \subset \overline{\mathcal{A}}^{wo}$, then there is a sequence (T_n) in $\mathcal{A}$ such that (T_n) weakly converges to T.*

Corollary 1.24. *Let X be a Banach space and let $\mathcal{A}$ be a relatively weakly compact subset of $\mathcal{B}(X)$. Then $\mathcal{A}$ is wo-closed (respectively w^*o-closed) if and only if $\mathcal{A}$ is wo-sequentially closed (respectively, w^*o-sequentially closed).*

Proof. Suppose that $\mathcal{A}$ is *wo*-sequentially closed. Let $T \in \overline{\mathcal{A}}^{wo}$. Then by Theorem 1.6 there is a sequence (T_n) in $\mathcal{A}$ such that (T_n) weakly converges to T. By (1.5) (T_n) *wo*-converges to T. Since $\mathcal{A}$ is *wo*-sequentially closed, $T \in \mathcal{A}$. Hence $\mathcal{A}$ is *wo*-closed. The proof of the another part of the corollary is similar. □

The proofs of above theorems are represented in Section 3, and in Section 4 we apply Theorem 1.8 to weak versions of the approximation property.

2. Some properties of *wo*, *w*o*, *sto*, and *τ*

We say that a subset B of a topological vector space is *bounded* with respect to that topology if, for each neighborhood U of 0, there is a $s_U > 0$ such that $B \subset tU$ whenever $t > s_U$. Note that if X is a topological vector space with a topology that is generated by a subspace $\mathcal{Z}$ of the vector space of all linear functionals on X, then for a subset B of X

$$\begin{aligned} &B \text{ is bounded with respect to that topology} \\ \Leftrightarrow \quad &\sup_{x\in B}\left|f(x)\right| \text{ is finite for each } f \in \mathcal{Z}. \end{aligned} \tag{2.1}$$

Proposition 2.1. *Let X be a Banach space and let $\mathcal{A}$ be a subset of $\mathcal{B}(X)$. Then the following are equivalent.*

(a) *$\mathcal{A}$ is bounded.*
(b) *$\mathcal{A}$ is τ-bounded.*
(c) *$\mathcal{A}$ is sto-bounded.*
(d) *$\mathcal{A}$ is weakly bounded.*
(e) *$\mathcal{A}$ is w^*o-bounded.*
(f) *$\mathcal{A}$ is wo-bounded.*

Proof. By (1.5) and (1.6) it is enough to show (f) ⇒ (a). Now let $x \in X$ and consider $\{Q_X(Tx)\colon T \in \mathcal{A}\}$, where Q_X is the natural map from X into X^{**}. Since $\mathcal{A}$ is *wo*-bounded, by (2.1) for each $x^* \in X^*$

$$\sup_{T\in\mathcal{A}}\left|Q_X(Tx)x^*\right| = \sup_{T\in\mathcal{A}}|x^*Tx| < \infty.$$

By the uniform boundedness principle

$$\sup_{T\in\mathcal{A}}\|Tx\| = \sup_{T\in\mathcal{A}}\left\|Q_X(Tx)\right\| < \infty.$$

Again by the uniform boundedness principle

$$\sup_{T\in\mathcal{A}}\|T\| < \infty.$$

Hence $\mathcal{A}$ is bounded. □

Proposition 2.2. *Let X be a Banach space. Then every wo-limit point compact subset $\mathcal{A}$ of $\mathcal{B}(X)$ is bounded.*

Proof. Note that by (2.1) a subset $\mathcal{A}$ of $\mathcal{B}(X)$ is *wo*-bounded if and only if $\sup_{T\in\mathcal{A}}|x^*Tx|$ is finite for each $x\in X$ and $x^*\in X^*$. Now suppose that $\mathcal{A}$ is unbounded. Then $\mathcal{A}$ is *wo*-unbounded by Proposition 2.1. So there are $x\in X$ and $x^*\in X^*$ such that $\sup_{T\in\mathcal{A}}|x^*Tx|$ is infinite. Then there is a sequence $(T_n)\subset\mathcal{A}$ such that $|x^*T_{n+1}x|\geqslant|x^*T_nx|+1$ for each n. Since $\mathcal{A}$ is *wo*-limit point compact, there is a $T\in\mathcal{A}$ such that T is a *wo*-limit point of $\{T_n\}$. Then a neighborhood $\{S\in\mathcal{B}(X)\colon |x^*Sx-x^*Tx|<1/2\}$ of T must contain elements T_{n_0}, T_{n_1} of $\{T_n\}$ with $n_0<n_1$. Thus the triangle inequality says

$$\left|x^*T_{n_1}x\right|-\left|x^*T_{n_0}x\right|<1.$$

This is a contradiction, which gives a proof of the proposition. □

By (1.5) and (1.6) we have the following corollary.

Corollary 2.3. *Let X be a Banach space. Then every weak limit (respectively, τ-limit, sto-limit, and w^*o-limit) point compact subset of $\mathcal{B}(X)$ is bounded.*

Lemma 2.4. *Let $\mathcal{F}$ be a finite-dimensional subspace of $\mathcal{B}(X)$ and $M>1$. Then there are finite subsets F_M, F_M^* of B_X and B_{X^*}, respectively, such that*

$$\|T\|\leqslant M\max\bigl\{|x^*Tx|\colon x\in F_M,\ x^*\in F_M^*\bigr\}$$

for each $T\in\mathcal{F}$.

Proof. Since the unit sphere $S_\mathcal{F}$ in $\mathcal{F}$ is compact, there is a subset $\{T_i\}_{i=1}^n$ of $S_\mathcal{F}$ such that

$$S_\mathcal{F}\subset\bigcup_{i=1}^n B\left(T_i;\frac{M-1}{2M}\right),$$

for each $B(T_i;\frac{M-1}{2M})$ an open ball with center T_i and radius $\frac{M-1}{2M}$. Since $(M+1)/2M<1$, there are subsets $\{x_i\}_{i=1}^n$, $\{x_i^*\}_{i=1}^n$ of B_X and B_{X^*}, respectively, such that

$$\left|x_i^*T_ix_i\right|>\frac{M+1}{2M}$$

for $i=1,\dots,n$. Let $F_M=\{x_i\}_{i=1}^n$ and $F_M^*=\{x_i^*\}_{i=1}^n$ and let $T\in S_\mathcal{F}$. Then there is a i such that

$$\|T-T_i\|<\frac{M-1}{2M}.$$

Now we have

$$\begin{aligned}\left|x_i^*Tx_i\right|&\geqslant\left|x_i^*T_ix_i\right|-\left|x_i^*T_ix_i-x_i^*Tx_i\right|\geqslant\left|x_i^*T_ix_i\right|-\|T_i-T\|\\&>\frac{M+1}{2M}-\frac{M-1}{2M}=\frac{1}{M}.\end{aligned}$$

So $\max\{|x^*Tx|\colon x\in F_M,\ x^*\in F_M^*\}>1/M$. By linearity we have

$$\|T\|\leqslant M\max\bigl\{|x^*Tx|\colon x\in F_M,\ x^*\in F_M^*\bigr\}$$

for each $T\in\mathcal{F}$. □

3. Proofs of the main theorems

Now we prove main theorems. The proofs of Theorems 1.2 and 1.8 are essentially the same. So we only prove Theorem 1.8.

Proof of Theorem 1.8. Let $\{x_n^*\}$ and $\{x_m^{**}\}$ be countable dense subsets of X^* and X^{**}, respectively. We may assume $x_n^* \neq 0$ and $x_m^{**} \neq 0$ for all n and m and let $y_n^* = x_n^*/\|x_n^*\|$ and $y_m^{**} = x_m^{**}/\|x_m^{**}\|$ for each n and m. Let the map $\varphi : \mathcal{A} \to \mathbf{F}^{\mathbf{N}\times\mathbf{N}}$ be defined by

$$\varphi(T) = \left(y_m^{**} T^* y_n^*\right)_{(y_n^*, y_m^{**}) \in \{y_n^*\} \times \{y_m^{**}\}}.$$

If $T, S \in \mathcal{A}$ with $T \neq S$, then $T^* \neq S^*$ and so there is a $x_0^* \in X^*$ such that $T^* x_0^* \neq S^* x_0^*$. Thus there is a $x_{n_0}^*$ such that $T^* x_{n_0}^* \neq S^* x_{n_0}^*$ since $\{x_n^*\}$ is dense in X^*. So there is a $x_0^{**} \in X^{**}$ such that $x_0^{**} T^* x_{n_0}^* \neq x_0^{**} S^* x_{n_0}^*$. Thus there is a $x_{m_0}^{**}$ such that $x_{m_0}^{**} T^* x_{n_0}^* \neq x_{m_0}^{**} S^* x_{n_0}^*$ since $\{x_m^{**}\}$ is dense in X^{**}. By linearity $y_{m_0}^{**} T^* y_{n_0}^* \neq y_{m_0}^{**} S^* y_{n_0}^*$. This shows that φ is one-to-one. Now let (T_α) be a net in $\mathcal{A}$ and $T \in \mathcal{A}$ and suppose $y_m^{**} T_\alpha^* y_n^* \to y_m^{**} T^* y_n^*$ for each n and m. Then by linearity $x_m^{**} T_\alpha^* x_n^* \to x_m^{**} T^* x_n^*$ for each n and m. Recall that $\mathcal{A}$ is bounded by Proposition 2.1. So let $M = \sup_{R \in \mathcal{A}} \|R\|$, and assume $x^* \in X^*$, $x^{**} \in X^{**}$ with $x^* \neq 0$ and $x^{**} \neq 0$, and $\epsilon > 0$. Choose $\delta > 0$ satisfying

$$\delta < \min\left\{\frac{\epsilon}{5M\|x^*\|}, \frac{\epsilon}{5M}\right\}.$$

Then there is a $x_{m_0}^{**} \in \{x_m^{**}\}$ such that $\|x^{**} - x_{m_0}^{**}\| < \delta$. Let $x_{n_0}^* \in \{x_n^*\}$ satisfies $\|x^* - x_{n_0}^*\| < \delta/\|x_{m_0}^{**}\|$. Since $x_m^{**} T_\alpha^* x_n^* \to x_m^{**} T^* x_n^*$ for each n and m, there is a β such that $\alpha \succeq \beta$ implies

$$\left|x_{m_0}^{**} T_\alpha^* x_{n_0}^* - x_{m_0}^{**} T^* x_{n_0}^*\right| < \frac{\epsilon}{5}.$$

Now by the triangle inequality $\alpha \succeq \beta$ implies

$$\begin{aligned}
&\left|x^{**} T_\alpha^* x^* - x^{**} T^* x^*\right| \\
&\quad \leqslant \left\|x^{**} - x_{m_0}^{**}\right\| \left\|T_\alpha^*\right\| \|x^*\| + \left\|x_{m_0}^{**}\right\| \left\|T_\alpha^*\right\| \left\|x^* - x_{n_0}^*\right\| + \left|x_{m_0}^{**} T_\alpha^* x_{n_0}^* - x_{m_0}^{**} T^* x_{n_0}^*\right| \\
&\qquad + \left\|x_{m_0}^{**}\right\| \left\|T^*\right\| \left\|x_{n_0}^* - x^*\right\| + \left\|x_{m_0}^{**} - x^{**}\right\| \left\|T^*\right\| \|x^*\| < \epsilon.
\end{aligned}$$

Thus $x^{**} T_\alpha^* x^* \to x^{**} T^* x^*$ for each $x^* \in X^*$ and $x^{**} \in X^{**}$. We have shown that $T_\alpha \to T$ in $(\mathcal{B}(X), w^*o)$ if and only if $\varphi(T_\alpha) \to \varphi(T)$ in the product topology. Hence φ is a w^*o-to-*pro* homeomorphism from ($\mathcal{A}$, the relative w^*o-topology of $\mathcal{A}$ in $\mathcal{B}(X)$) onto ($\varphi(\mathcal{A})$, the relative *pro*-topology of $\varphi(\mathcal{A})$ in $\mathbf{F}^{\mathbf{N}\times\mathbf{N}}$), where *pro* means the product topology. Note that $\mathbf{F}^{\mathbf{N}\times\mathbf{N}}$ satisfies the *pro*-second countability axiom since $\mathbf{F}^{\mathbf{N}\times\mathbf{N}}$ is a countable product of spaces satisfying the second countability axiom. Thus the relative *pro*-topology of $\varphi(\mathcal{A})$ has the second countability axiom. Hence the relative w^*o-topology of $\mathcal{A}$ has the second countability axiom.

Now for $T, S \in \mathcal{A}$ consider

$$d(T, S) = \sum_{(y_n^*, y_m^{**}) \in \{y_n^*\} \times \{y_m^{**}\}} \frac{|y_m^{**}(T - S)^* y_n^*|}{2^{n+m}}.$$

Since φ is one-to-one, $d(T, S) = 0$ if and only if $T = S$. Clearly $d(T, R) \leqslant d(T, S) + d(S, R)$ for $T, S, R \in \mathcal{A}$. Thus d is a metric, moreover, for $\alpha \in \mathbf{F}$ $d(\alpha T, \alpha S) = |\alpha| d(T, S)$.

Let (T_α) be a net in $\mathcal{A}$ and $T \in \mathcal{A}$. Recall that $x^{**}T_\alpha^* x^* \to x^{**}T^* x^*$ for each $x^* \in X^*$ and $x^{**} \in X^{**}$ if and only if $y_m^{**}T_\alpha^* y_n^* \to y_m^{**}T^* y_n^*$ for each n and m. Since $\mathcal{A}$ is bounded, we can check that $T_\alpha \to T$ in ($\mathcal{B}(X)$, the relative w^*o-topology of $\mathcal{A}$) if and only if $d(T_\alpha, T) \to 0$. Hence the relative w^*o-topology of $\mathcal{A}$ is induced by the metric d. $\square$

Proof of Theorem 1.13. Let $\{x_n\}$ be a countable dense subset of X. We may assume $x_n \neq 0$ for each n and let $y_n = x_n/\|x_n\|$ for each n. Let the map $\varphi : \mathcal{A} \to X^{\mathbf{N}}$ be defined by

$$\varphi(T) = (Ty_n)_{y_n \in \{y_n\}}.$$

As in the proof of Theorem 1.8, we can check that φ is a *sto*-to-*pro* homeomorphism from ($\mathcal{A}$, the relative *sto*-topology of $\mathcal{A}$ in $\mathcal{B}(X)$) onto ($\varphi(\mathcal{A})$, the relative *pro*-topology of $\varphi(\mathcal{A})$ in $X^{\mathbf{N}}$), where *pro* means the product topology. Since X is a separable metric space, X satisfies the second countability axiom. Thus $X^{\mathbf{N}}$ satisfies the *pro*-second countability axiom since $X^{\mathbf{N}}$ is a countable product of spaces satisfying the second countability axiom. Thus the relative *pro*-topology of $\varphi(\mathcal{A})$ has the second countability axiom. Hence the relative *sto*-topology of $\mathcal{A}$ has the second countability axiom.

Now for $T, S \in \mathcal{A}$ consider

$$d(T, S) = \sum_n \frac{\|(T - S)y_n\|}{2^n}.$$

Then as in the proof of Theorem 1.8, d is a metric and for $\alpha \in \mathbf{F}$ $d(\alpha T, \alpha S) = |\alpha| d(T, S)$, and for a net (T_α) in $\mathcal{A}$, $T \in \mathcal{A}$, we can check that $T_\alpha \to T$ in ($\mathcal{B}(X)$, the relative *sto*-topology of $\mathcal{A}$) if and only if $d(T_\alpha, T) \to 0$. Hence the relative *sto*-topology of $\mathcal{A}$ is induced by the metric d. $\square$

Proof of Theorem 1.18. By Proposition 2.1 it is enough to show that the relative τ-topology and the relative strong operator topology of bounded subsets of $\mathcal{B}(X)$ are the same. Now let $\mathcal{A}$ be a bounded subset of $\mathcal{B}(X)$. Since $sto \leqslant \tau$, we only show $\tau \leqslant sto$ on $\mathcal{A}$. Since $\mathcal{A}$ is bounded, $\sup_{T \in \mathcal{A}} \|T\| \leqslant \lambda$ for some $\lambda > 0$. Let $(T_\alpha) \subset \mathcal{A}$ be a net and $T \in \mathcal{A}$ with $T_\alpha \to T$ in $(\mathcal{B}(X), sto)$. Let $K \subset X$ a compact and $\epsilon > 0$. Then there is a finite $F \subset K$ such that whenever $x \in K$ we have

$$\|x - y\| < \frac{\epsilon}{3\lambda}$$

for some $y \in F$. Since $T_\alpha \to T$ in $(\mathcal{B}(X), sto)$, there is a β such that $\alpha \succeq \beta$ implies $\|T_\alpha y - Ty\| < \epsilon/3$ for every $y \in F$. Now let $x \in K$. Then there is a $y \in F$ such that $\|x - y\| < \epsilon/3\lambda$. Thus by the triangle inequality $\alpha \succeq \beta$ implies

$$\|T_\alpha x - Tx\| < \epsilon.$$

Hence $T_\alpha \to T$ in $(\mathcal{B}(X), \tau)$. This completes the proof. $\square$

The proof of Theorem 1.6 is essentially the proof of Day's lemma ([5, Theorem 3.2.4], [14, Lemma 2.8.5]).

Proof of Theorem 1.6. We may assume $T \notin \mathcal{A}$ and considering $-T + \mathcal{A}$, then we may assume $T = 0$. Now we show the following statement. For each n there are $\{T_i\}_{i=1}^{n+1} \subset \mathcal{A}$, finite subsets F_n, F_n^* of B_X and B_{X^*}, respectively, with $F_n \subset F_{n+1}$, $F_n^* \subset F_{n+1}^*$ such that the following hold:

$$\|T\| \leqslant 2\max\{|x^*Tx|: x \in F_n, x^* \in F_n^*\} \quad \text{for each } T \in \langle \{T_i\}_{i=1}^n \rangle, \tag{3.1}$$

$$\max\{|x^*T_{n+1}x|: x \in F_n, x^* \in F_n^*\} < \frac{1}{n+1}. \tag{3.2}$$

Let $T_1 \in \mathcal{A}$. Then there are $x_1 \in B_X$ and $x_1^* \in B_{X^*}$ such that

$$|x_1^*T_1x_1| \geqslant \frac{1}{2}\|T_1\|.$$

Let $F_1 = \{x_1\}$ and $F_1^* = \{x_1^*\}$. Then (3.1) holds. Since $\{T \in \mathcal{B}(X): |x_1^*Tx_1| < \frac{1}{2}\}$ is a *wo*-neighborhood of 0, there is a $T_2 \in \mathcal{A} \backslash \{T_1\}$ such that

$$T_2 \in \left\{T \in \mathcal{B}(X): |x_1^*Tx_1| < \frac{1}{2}\right\}.$$

So (3.2) holds. Now suppose that there are finite subsets F_n, F_n^* of B_X and B_{X^*}, respectively, and $\{T_i\}_{i=1}^{n+1} \subset \mathcal{A}$ such that (3.1) and (3.2) hold. By Lemma 2.4 there are finite subsets $F_n{}'$, $F_n^{*\prime}$ of B_X and B_{X^*}, respectively, such that $\|T\| \leqslant 2\max\{|x^*Tx|: x \in F_n{}', x^* \in F_n^{*\prime}\}$ for each $T \in \langle \{T_i\}_{i=1}^{n+1} \rangle$. Let $F_{n+1} = F_n \cup F_n{}'$ and $F_{n+1}^* = F_n^* \cup F_n^{*\prime}$. Then we have

$$\|T\| \leqslant 2\max\{|x^*Tx|: x \in F_{n+1}, x^* \in F_{n+1}^*\}$$

for each $T \in \langle \{T_i\}_{i=1}^{n+1} \rangle$. Thus (3.1) holds, $F_n \subset F_{n+1}$, and $F_n^* \subset F_{n+1}^*$. Since $\{T \in \mathcal{B}(X): |x^*Tx| < 1/(n+2),\ x \in F_{n+1}, x^* \in F_{n+1}^*\}$ is a *wo*-neighborhood of 0, there is a $T_{n+2} \in \mathcal{A} \setminus \{T_i\}_{i=1}^{n+1}$ such that

$$T_{n+2} \in \left\{T \in \mathcal{B}(X): |x^*Tx| < \frac{1}{n+2},\ x \in F_{n+1}, x^* \in F_{n+1}^*\right\}.$$

Thus (3.2) holds. By induction, for all n (3.1) and (3.2) hold, and $F_n \subset F_{n+1}$, and $F_n^* \subset F_{n+1}^*$. Let $D = \bigcup_n F_n$ and $D^* = \bigcup_n F_n^*$. Then $\|T\| \leqslant 2\sup\{|x^*Tx|: x \in D, x^* \in D^*\}$ for each $T \in \langle \{T_n\} \rangle$. Thus

$$\|T\| \leqslant 2\sup\{|x^*Tx|: x \in D, x^* \in D^*\}$$

for each $T \in [\{T_n\}]$. Since $\mathcal{A}$ is relatively weakly compact, also $\mathcal{A}$ is relatively weakly limit point compact. Thus $\{T_n\}$ has a weak limit point S. Since $S \in \overline{\{T_n\}}^{weak} \subset \overline{\langle \{T_n\} \rangle}^{weak} = [\{T_n\}]$, $\|S\| \leqslant 2\sup\{|x^*Sx|: x \in D, x^* \in D^*\}$. Now let $x_0 \in D$, $x_0^* \in D^*$, and $\epsilon > 0$. Since $S \in \overline{\{T_n\}}^{weak} \subset \overline{\{T_n\}}^{wo}$, $\{T \in \mathcal{B}(X): |x_0^*Tx_0 - x_0^*Sx_0| < \epsilon/2\}$ contain infinitely many members of $\{T_n\}$. By (3.2) there is a n_0 such that

$$\max\{|x^*T_{n_0+1}x|: x \in F_{n_0}, x^* \in F_{n_0}^*\} < \frac{\epsilon}{2}, \quad x_0 \in F_{n_0},\ x_0^* \in F_{n_0}^*, \quad \text{and}$$

$$T_{n_0+1} \in \left\{T \in \mathcal{B}(X): |x_0^*Tx_0 - x_0^*Sx_0| < \frac{\epsilon}{2}\right\}.$$

Thus $|x_0^*Sx_0| < \epsilon$ by the triangle inequality. Since ϵ is arbitrary then $x_0^*Sx_0 = 0$, that is, $x^*Sx = 0$ for all $x \in D$ and $x^* \in D^*$. Since $\|S\| \leqslant 2\sup\{|x^*Sx|: x \in D, x^* \in D^*\}$, $S = 0$.

Thus $\{T_n\}$ has only the zero weak limit point. Suppose that (T_n) does not weakly converge to 0. Then there is a weak neighborhood $\mathcal{U}$ of 0 and subsequence (T_{n_k}) of (T_n) such that $(T_{n_k}) \subset \mathcal{B}(X) \setminus \mathcal{U}$. Since $\mathcal{A}$ is relatively weakly limit point compact, (T_{n_k}) has a weak limit point R but $R \neq 0$. Since $(T_{n_k}) \subset (T_n)$, R is a weak limit point of (T_n). So R must be zero, which is a contradiction. Thus (T_n) weakly converges to 0. Hence (T_n) is a desired sequence, which proves the theorem. $\square$

4. Applications

Throughout this section, we will use the following notations:

$\mathcal{F}(X)$: The collection of bounded and finite rank linear operators on X.
$\mathcal{F}(X,\lambda)$: The collection of bounded and finite rank linear operators on X satisfying $\|T\| \leqslant \lambda$.
$\mathcal{K}(X)$: The collection of compact operators on X.

We say that a Banach space X has the *approximation property* (in short, AP) if for every compact set $K \subset X$ and every $\epsilon > 0$, there is a $T \in \mathcal{F}(X)$ such that $\|Tx - x\| < \epsilon$ for all $x \in K$.

Grothendieck [9] showed the following characterization of the AP:

X has the AP iff for every Banach space Y and every compact operator T from Y into X there is a sequence (T_n) of finite rank operators from Y into X such that

$$\|T_n - T\| \to 0.$$

For various results of the approximation property and other approximation properties one may see Casazza [3], Lindenstrauss and Tzafriri [13]. Recently Choi and Kim [4] introduced weak versions of the approximation property.

We say that X has the *quasi approximation property* (in short, QAP) if for every $T \in \mathcal{K}(X)$ there is a sequence (T_n) in $\mathcal{F}(X)$ such that $\|T_n - T\| \to 0$. Also we say that X has the *bounded weak approximation property* (in short, BWAP) if for every $T \in \mathcal{K}(X)$, for some $\lambda_T > 0$ there is a net (T_α) in $\mathcal{F}(X, \lambda_T)$ such that $T_\alpha \to T$ in $(\mathcal{B}(X), \tau)$. Thus by the characterization of the AP we have the following implication:

$$\text{AP} \quad \Rightarrow \quad \text{QAP} \quad \Rightarrow \quad \text{BWAP}.$$

It is well known that if X^* has the AP, then X has the AP [13] and in [4] it was shown that if X^* has the BWAP, then X has the BWAP. For the QAP we have the following result.

Theorem 4.1. *Let X be a Banach space such that X^{**} is separable. If X^* has the BWAP, then X has the QAP.*

Since the QAP implies the BWAP, we have the following corollary.

Corollary 4.2. *Let X be a Banach space such that X^{**} is separable. If X^* has the QAP, then X has the QAP.*

The proof of Theorem 4.1 is based on the following two lemmas and Theorem 1.8.

Lemma 4.3. [11] *Let X be a Banach space. If $T_n \to T$ in $(\mathcal{B}(X), w^*o)$, where (T_n) is a sequence in $\mathcal{K}(X)$ and $T \in \mathcal{K}(X)$, then there is a sequence (S_n) of convex combinations of $\{T_n\}$ such that $\|S_n - T\| \to 0$.*

Lemma 4.4. [12] *Let X be a Banach space. If X^* has the BWAP, then for every $T \in \mathcal{K}(X)$, for some $\lambda_T > 0$ there is a net (T_α) in $\mathcal{F}(X, \lambda_T)$ such that $T_\alpha \to T$ in $(\mathcal{B}(X), w^*o)$.*

Proof of Theorem 4.1. Suppose that X^* has the BWAP and let $T \in \mathcal{K}(X)$. Then by Lemma 4.4 there is a $\lambda_T > 0$ such that $T \in \overline{\mathcal{F}(X, \lambda_T)}^{w^*o}$. Now Theorem 1.8 says that there is a sequence (T_n) in $\mathcal{F}(X, \lambda_T)$ such that $T_n \to T$ in $(\mathcal{B}(X), w^*o)$. Since $\mathcal{F}(X) \subset \mathcal{K}(X)$, Lemma 4.3 says that there is a sequence (S_n) in $\mathcal{F}(X)$ (moreover, $\mathcal{F}(X, \lambda_T)$) such that $\|S_n - T\| \to 0$. Hence X has the QAP. $\square$

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